

Thermodynamics of Classical Systems on Noncommutative Phase Space

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We study the formulation of statistical mechanics on noncommutative classical phase space. We construct canonical ensemble theory in noncommutative phase spaces. We consider for illustration some basic and important examples in the framework of noncommutative statistical mechanics such as Ideal Gas, Extreme Relativistic Gas and 3-Dimensional Harmonic Oscillator.

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1. Introduction

In recent years, there has been an increasing interest in the study of physics on noncommutative space, because the effects of the space noncommutativity may become significant in the extreme situation such as at the string scale or at the TeV and even higher energy level. There are many papers devoted to the study of various aspects of the quantum field theory and quantum mechanics on NC space, where space-space is noncommuting, but the momentum-momentum is commuting, or on NC phase space, where both space-space and momentum-momentum are noncommuting. For references see [1]-[5]. The Bose-Einstein statistics on noncommutative quantum

mechanics requires both space-space and momentum-momentum noncommutativity [6]. On NC phase space, we can consider the NC algebra as

$$[\hat{x}_i, \hat{x}_j] = i\hbar\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\hbar\bar{\theta}_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar(\delta_{ij} - \frac{1}{4}\theta_{ik}\bar{\theta}_{kj}), \quad (1)$$

where θ_{ij} is related to the noncommutativity of the space, $\bar{\theta}_{ij}$ reflects the noncommutativity of the momenta and both of them are the antisymmetric matrices with real constant elements. From the above relations, we can obtain a generalized Bopp's shift as

$$\hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j, \quad (2)$$

$$\hat{p}_i = p_i + \frac{1}{2}\bar{\theta}_{ij}x_j, \quad (3)$$

where x_i and p_i are the coordinate and momentum operators on usual (commutative) phase space, and $i, j = 1, 2, 3$. After applying this shift, the effect caused by phase space noncommutativity can be calculated in the usual phase space [7]. In NC quantum mechanics and NC quantum field theory, the star product between two fields on NC phase space can be replaced by the generalized Bopp's shift (2) for coordinates and (3) for momenta. The star product on NC phase space can be defined as [7]

$$\begin{aligned} (f * g)(x, p) &= f(x, p) e^{\frac{i\hbar}{2}\overleftarrow{\partial}_i^x \theta_{ij} \overrightarrow{\partial}_j^x + \frac{i\hbar}{2}\overleftarrow{\partial}_i^p \bar{\theta}_{ij} \overrightarrow{\partial}_j^p} g(x, p) \\ &= f(x, p)g(x, p) + \frac{i\hbar}{2}\theta_{ij}\partial_i^x f(x, p)\partial_j^x g(x, p) \\ &\quad + \frac{i\hbar}{2}\bar{\theta}_{ij}\partial_i^p f(x, p)\partial_j^p g(x, p) + O(\theta^2) + O(\bar{\theta}^2). \end{aligned} \quad (4)$$

In this work, in Section 2, we present the uncertainty relation on NC phase space to obtain the deformation of Planck constant in order to make the classical partition function dimensionless. In Section 3, we will assume that we have a symplectic structure consistent with the commutation rules (1) and we will obtain the corresponding classical partition function. In Sections 4 to 6, we will see three concrete examples, the classical ideal gas, the extreme relativistic gas and the classical harmonic oscillator in a 3-dimensional NC phase space and the new features that arise.

2. The Uncertainty Relation on NC phase space

In 3-dimensional classical partition function, for a single particle, we put $\frac{1}{h^3}$ as the quantity which makes the volume of phase space dimensionless.

In this section, we want to find the factor that makes the volume of NC phase space dimensionless. From the relation (1), we can write

$$[\hat{x}_i, \hat{p}_j] = i\tilde{h}_{ij}, \quad (5)$$

where \tilde{h}_{ij} as a tensor plays the role of the deformation of Planck constant on NC phase space. If we put $\theta_3 = \theta$ and $\bar{\theta}_3 = \bar{\theta}$ and the rest of the θ and $\bar{\theta}$ components to zero (which can be done by a rotation or a redefinition of coordinates), we can rewrite Eq. (5) as

$$[\hat{x}_i, \hat{p}_j] = i\hbar \begin{pmatrix} 1 + \frac{\theta\bar{\theta}}{16} & 0 & 0 \\ 0 & 1 + \frac{\theta\bar{\theta}}{16} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad i, j = 1, 2, 3 \quad (6)$$

where $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k$ and $\bar{\theta}_{ij} = \frac{1}{2}\epsilon_{ijk}\bar{\theta}_k$. If we consider the following relations

$$[\hat{x}_1, \hat{p}_1] = i\tilde{h}_{11} = i\hbar \left(1 + \frac{\theta\bar{\theta}}{16}\right), \quad (7)$$

$$[\hat{x}_2, \hat{p}_2] = i\tilde{h}_{22} = i\hbar \left(1 + \frac{\theta\bar{\theta}}{16}\right), \quad (8)$$

$$[\hat{x}_3, \hat{p}_3] = i\tilde{h}_{33} = i\hbar, \quad (9)$$

it leads to the following uncertainty relations on NC phase space

$$\Delta\hat{x}_1\Delta\hat{p}_1 \sim \tilde{h}_{11} = h \left(1 + \frac{\theta\bar{\theta}}{16}\right), \quad (10)$$

$$\Delta\hat{x}_2\Delta\hat{p}_2 \sim \tilde{h}_{22} = h \left(1 + \frac{\theta\bar{\theta}}{16}\right), \quad (11)$$

$$\Delta\hat{x}_3\Delta\hat{p}_3 \sim \tilde{h}_{33} = h. \quad (12)$$

From the above relations, it is obvious that the following factor makes the volume of NC phase space dimensionless

$$\frac{1}{\tilde{h}_{11}\tilde{h}_{22}\tilde{h}_{33}} = \frac{1}{\hbar^3} = \frac{1}{h^3(1 + \frac{\theta\bar{\theta}}{8})}. \quad (13)$$

In this place, we considered the denominator up to second order of NC parameters. It should be mentioned the deformation of Planck constant, on 2-dimentional NC phase space, has the form [7]

$$\tilde{h} = h \left(1 + \frac{\theta\bar{\theta}}{4}\right), \quad (14)$$

where $\theta_{ij} = \epsilon_{ij}\theta$ and $\bar{\theta}_{ij} = \epsilon_{ij}\bar{\theta}$. Thus, the factor that makes the volume of NC phase space dimensionless, is

$$\frac{1}{\tilde{h}^2} = \frac{1}{h^2(1 + \frac{\theta\bar{\theta}}{2})}. \quad (15)$$

3. Classical Partition Function on NC Phase Space

The purpose of this paper is precisely to study the noncommutative classical systems. The passage between NC classical mechanics and NC quantum mechanics is assumed to be realized via the following generalized Dirac quantization:

$$\{f, g\} \longrightarrow \frac{1}{i\hbar}[O_f, O_g] \quad (16)$$

where we denote by O_f the operator associated to a classical observable f . This quantization generalizes the relations (1) in the following way:

$$\{\hat{q}_i, \hat{q}_j\} = \theta_{ij}, \quad \{\hat{p}_i, \hat{p}_j\} = \bar{\theta}_{ij}, \quad \{\hat{q}_i, \hat{p}_j\} = \delta_{ij} - \frac{1}{4}\theta_{ik}\bar{\theta}_{kj}, \quad (17)$$

where \hat{q}_i and \hat{p}_i are the coordinate and momentum classical observables in NC phase space. Moreover, θ_{ij} and $\bar{\theta}_{ij}$ are of dimension of $(length)^2/\hbar$ and $(momentum)^2/\hbar$ respectively, and $i, j = 1, 2, 3$. From the above relations, we can obtain NC classical observables as

$$\hat{q}_i = q_i - \frac{1}{2}\theta_{ij}p_j, \quad (18)$$

$$\hat{p}_i = p_i + \frac{1}{2}\bar{\theta}_{ij}q_j, \quad (19)$$

where $q_i = (x, y, z)$ and $p_i = (p_x, p_y, p_z)$ are the classical observables on usual (commutative) phase space. We can see, in the classical limit, the symplectic structure (17) will not have \hbar , as should be.

To obtain the classical partition function, in canonical ensemble, on NC phase space, we can consider the following formula

$$Q_1^{NC} = \frac{1}{\tilde{h}^3} \int e^{-\beta H^{NC}(\hat{q}, \hat{p})} d^3\hat{q} d^3\hat{p}, \quad (20)$$

where is written for a single particle. $\frac{1}{\tilde{h}^3}$ was obtained in Section 2. $H^{NC}(\hat{q}, \hat{p})$ is Hamiltonian of a NC classical system, in terms of noncanonical coordinates and momenta. According to the relations (18) and (19), and considering $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k$ and $\bar{\theta}_{ij} = \frac{1}{2}\epsilon_{ijk}\bar{\theta}_k$, it is easy to write the integration measure in (20), up to second order of NC parameters, as

$$d^3\hat{q} d^3\hat{p} = (1 - \frac{\theta\bar{\theta}}{8}) d^3q d^3p, \quad (21)$$

where we considered $\vec{\theta} = (0, 0, \theta)$ and $\vec{\bar{\theta}} = (0, 0, \bar{\theta})$. If we write $H^{NC}(\hat{q}, \hat{p})$ in terms of canonical coordinates and momenta (which can be done by applying (18) and (19) to the Hamiltonian), the appropriate expression for the partition function would, therefore, be

$$\begin{aligned} Q_1^{NC} &= \frac{1}{h^3(1 + \frac{\theta\bar{\theta}}{8})} \int e^{-\beta H^{NC}(q,p)} \left(1 - \frac{\theta\bar{\theta}}{8}\right) d^3q d^3p \\ &= \frac{1}{h^3} \int e^{-\beta H^{NC}(q,p)} \left(1 - \frac{\theta\bar{\theta}}{4}\right) d^3q d^3p . \end{aligned} \quad (22)$$

where we used (13) and (21). $H^{NC}(q, p)$ is the Hamiltonian of a NC classical system, in terms of canonical coordinates and momenta. Thus, to find the partition function of a single particle of a NC classical system, it is enough to rewrite the Hamiltonian of the system in terms of canonical classical observables and put it into relation (22). If we consider that the basic constituents of our system are non-interacting, we can write the classical partition function for a system of N particles on a 3-dimensional NC phase space as

$$Q_N^{NC} = \frac{1}{N!} [Q_1^{NC}]^N, \quad (23)$$

where $1/N!$ is Gibbs correction factor.

4. Classical Ideal Gas on NC Phase Space

Let us consider a system of N identical molecules, assumed to be monatomic (so that there are no internal degrees of motion to be considered), confined to a space of volume $V (= L^3)$ and in equilibrium at temperature T. Since there are no intermolecular interactions to be taken into account, the Hamiltonian of a single molecule of the system, on NC classical phase space, is

$$H^{NC}(\hat{q}, \hat{p}) = \frac{\hat{p}_i \hat{p}_i}{2m} . \quad (24)$$

Thus, from the relations (18) and (19), the Hamiltonian takes the following form

$$H^{NC}(q, p) = \frac{1}{2m} \left(p^2 - \frac{1}{2} \bar{\theta} L_z + \frac{1}{16} \bar{\theta}^2 (x^2 + y^2) \right) , \quad (25)$$

where we considered $\vec{\theta} \cdot (\vec{q} \times \vec{p}) = \bar{\theta} L_z$ and $(\vec{\theta} \times \vec{q})^2 = \bar{\theta}^2 (x^2 + y^2)$, by assumption

$$\vec{\theta} = (0, 0, \theta) , \quad \theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k \quad (26)$$

and

$$\vec{\theta} = (0, 0, \bar{\theta}) , \quad \bar{\theta}_{ij} = \frac{1}{2} \epsilon_{ijk} \bar{\theta}_k . \quad (27)$$

The partition function would then be

$$Q_1^{NC} = \frac{1}{h^3} \int e^{-\beta \frac{1}{2m} (p^2 - \frac{1}{2} \bar{\theta} L_z + \frac{1}{16} \bar{\theta}^2 (x^2 + y^2))} \left(1 - \frac{\theta \bar{\theta}}{4}\right) d^3 q d^3 p . \quad (28)$$

By calculating the integration, up to second order of NC parameters, and using the Eq. (23), we reach to

$$Q_N^{NC} = \frac{V^N}{N! h^3} (2\pi m K T)^{\frac{3N}{2}} \left(1 - \frac{\theta \bar{\theta}}{4}\right)^N . \quad (29)$$

Now the Helmholtz free energy is then given by

$$A^{NC} = -K T \ln(Q_N^{NC}) = A + N K T \theta \bar{\theta} / 4 , \quad (30)$$

where A is the Helmholtz free energy in ordinary (commutative) phase space. The complete thermodynamics of the ideal gas can be derived from (29) and (30) in a straightforward manner. For instance,

$$S^{NC} = - \left(\frac{\partial A^{NC}}{\partial T} \right)_{N,V} = S - N K \theta \bar{\theta} / 4 , \quad (31)$$

$$\mu^{NC} = \left(\frac{\partial A^{NC}}{\partial N} \right)_{V,T} = \mu + K T \theta \bar{\theta} / 4 , \quad (32)$$

$$P^{NC} = - \left(\frac{\partial A^{NC}}{\partial V} \right)_{N,T} = P , \quad (33)$$

$$U^{NC} = - \frac{\partial}{\partial \beta} \ln(Q_N^{NC}) = U , \quad (34)$$

$$C_V^{NC} = \left(\frac{\partial U^{NC}}{\partial T} \right)_{N,V} = C_V , \quad (35)$$

where S , μ , P , U and C_V are usual thermodynamic quantities which have been calculated in [8].

5. Extreme Relativistic Gas on NC Phase-Space

Let us consider an ideal extreme relativistic gas consisting of N monatomic molecules with energy-momentum relationship $E = pc$, c being the speed of

light, confined to a space of volume $V(=L^3)$ and in equilibrium at temperature T . On NC classical phase space, the Hamiltonian of a single molecule of the system is

$$\begin{aligned} H^{NC}(q, p) &= c\sqrt{\hat{p}_i\hat{p}_i} \\ &= c\sqrt{\left(p_i + \frac{1}{2}\bar{\theta}_{ij}q_j\right)\left(p_i + \frac{1}{2}\bar{\theta}_{ik}q_k\right)} \\ &= pc - \frac{c\bar{\theta}L_z}{4p} + \frac{c\bar{\theta}^2}{32p}(x^2 + y^2) - \frac{c\bar{\theta}^2L_z^2}{32p^3} + O(\bar{\theta}^3), \end{aligned} \quad (36)$$

where we used the conditions of (26) and (27). Regarding (22), we readily obtain the partition function as

$$\begin{aligned} Q_1^{NC} &= \frac{1}{h^3} \int e^{-\beta\left(pc - \frac{c\bar{\theta}L_z}{4p} + \frac{c\bar{\theta}^2}{32p}(x^2 + y^2) - \frac{c\bar{\theta}^2L_z^2}{32p^3}\right)} \left(1 - \frac{\theta\bar{\theta}}{4}\right) d^3q d^3p \\ &= 8\pi V \left(\frac{KT}{hc}\right)^3 \left(1 - \frac{\theta\bar{\theta}}{4}\right), \end{aligned} \quad (37)$$

whence

$$Q_N^{NC} = \frac{1}{N!} (8\pi V)^N \left(\frac{KT}{hc}\right)^{3N} \left(1 - \frac{\theta\bar{\theta}}{4}\right)^N. \quad (38)$$

Finally, the complete thermodynamics of the ideal extreme relativistic gas, on NC classical phase space, would be similar to (30) to (35).

6. 3-Dimentional Harmonic Oscillator on NC Phase Space

We shall now examine a system of N , practically independent, harmonic oscillators. The Hamiltonian of any one of them (assumed to be 3-dimensional), up to second order, is then given by

$$H^{NC}(\hat{q}, \hat{p}) = \frac{1}{2m}(\hat{p}_i\hat{p}_i) + \frac{1}{2}m\omega^2(\hat{q}_i\hat{q}_i), \quad (39)$$

or

$$\begin{aligned} H^{NC}(q, p) &= \frac{1}{2m}(p_i + \frac{1}{2}\bar{\theta}_{ij}q_j)(p_i + \frac{1}{2}\bar{\theta}_{ik}q_k) + \frac{1}{2}m\omega^2(q_i - \frac{1}{2}\theta_{ij}p_j)(q_i - \frac{1}{2}\theta_{ik}p_k) \\ &= \frac{1}{2m}\left(p^2 - \frac{1}{2}\bar{\theta}L_z + \frac{1}{16}\bar{\theta}^2(x^2 + y^2)\right) \\ &\quad + \frac{1}{2}m\omega^2\left(q^2 - \frac{1}{2}\theta L_z + \frac{1}{16}\theta^2(p_x^2 + p_y^2)\right), \end{aligned} \quad (40)$$

where we applied (26) and (27). Thus, regarding (22), for the single-oscillator, we readily obtain the partition function as

$$Q_1^{NC} = \frac{1}{h^3} \int e^{-\frac{\beta}{2m}(p^2 - \frac{1}{2}\bar{\theta}L_z + \frac{1}{16}\bar{\theta}^2(x^2+y^2)) - \frac{\beta}{2}m\omega^2(q^2 - \frac{1}{2}\theta L_z + \frac{1}{16}\theta^2(p_x^2+p_y^2))} \times (1 - \frac{\theta\bar{\theta}}{4}) d^3q d^3p \quad (41)$$

$$= \frac{1}{(\beta\hbar\omega)^3} \frac{1}{(1 - \frac{\theta\bar{\theta}}{8})} (1 - \frac{\theta\bar{\theta}}{4}) \quad (42)$$

$$= \frac{1}{(\beta\hbar\omega)^3} (1 - \frac{\theta\bar{\theta}}{8}), \quad (43)$$

whence

$$Q_N^{NC} = \frac{1}{(\beta\hbar\omega)^{3N}} \left(1 - \frac{\theta\bar{\theta}}{8}\right)^N. \quad (44)$$

Therefore, the complete thermodynamics of the 3-dimentional harmonic oscillator will be as following

$$A^{NC} = -KT \ln(Q_N^{NC}) = A + NKT\theta\bar{\theta}/8, \quad (45)$$

$$S^{NC} = -\left(\frac{\partial A^{NC}}{\partial T}\right)_{N,V} = S - NK\theta\bar{\theta}/8, \quad (46)$$

$$\mu^{NC} = \left(\frac{\partial A^{NC}}{\partial N}\right)_{V,T} = \mu + KT\theta\bar{\theta}/8, \quad (47)$$

$$P^{NC} = -\left(\frac{\partial A^{NC}}{\partial V}\right)_{N,T} = P, \quad (48)$$

$$U^{NC} = -\frac{\partial}{\partial \beta} \ln(Q_N^{NC}) = U, \quad (49)$$

$$C_V^{NC} = \left(\frac{\partial U^{NC}}{\partial T}\right)_{N,V} = C_V. \quad (50)$$

7. Conclusions

We have studied the effects of NC phase space on some classical statistical systems. We presented a formulation to calculate the classical partition function according to the canonical ensemble theory up to second order of noncommutative parameters (22). To make the volume of NC phase space dimensionless, we used the uncertainly relations ((10) - (12)) and, then,

found (13) and (15) for the three and two dimensional NC phase space . In Sections 4, 5 and 6, we considered the three well-known classical system on NC phase space. The results show that the terms arising from noncommutativity of phase space emerge in thermodynamic quantities that are not measurable (A , S and μ). We conclude that, in classic statistical mechanics, the present experimental instruments cannot currently detect the effects of noncommutativity, at least to second order of NC parameters. The final point that must be emphasized is about entropy. We know that NC phase space is more ordered than commutative phase space. The reason is that, when you are solving a problem on NC phase space, the degeneracy of the system decreases [9]. Therefore, we expect that the entropy of the system reduces in comparison with commutative phase space. This foreknowledge has taken place in relations (31) and (46).

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